Spectral characterizations of graphs

Willem H Haemers

Tilburg University

The Netherlands

Which structural properties of a graph are determined by the eigenvalues of the adjacency or the Laplacian matrix?

Willem H Haemers

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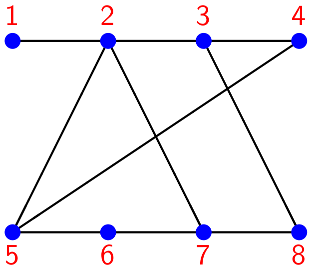
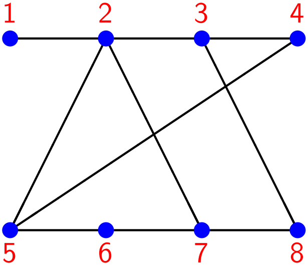
$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



adjacency spectrum

$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

adjacency spectrum

$$\{-1-\sqrt{3}\,,\;\; -1\,,\;\; -1\,,\;\; 1-\sqrt{3}\,,\;\; -1+\sqrt{3}\,,\;\; 1\,,\;\; 1\,,\;\; 1+\sqrt{3}\}$$

adjacency spectrum

$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

The adjacency spectrum is symmetric around 0 adjacency spectrum

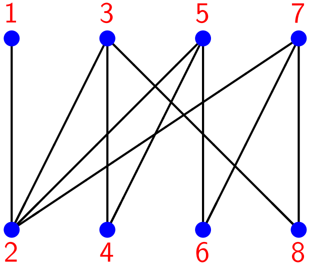
$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

Theorem (Coulson, Rushbrooke 1940, Sachs 1966)

The adjacency spectrum is symmetric around 0

if and only if the graph is bipartite

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$



adjacency spectrum

$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

$\lambda_1 \geq \ldots \geq \lambda_n$ are the adjacency eigenvalues of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is regular if and only if λ_1 equals the average degree

$\lambda_1 \geq \ldots \geq \lambda_n$ are the adjacency eigenvalues of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is regular if and only if λ_1 equals $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$

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Theorem

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Theorem

G is regular if and only if λ_1 equals $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$

Drawback

Spectrum does not tell everything

$$\{-1-\sqrt{3}\ ,\quad -1\ ,\quad -1\ ,\quad 1-\sqrt{3}\ ,\quad -1+\sqrt{3}\ ,\quad 1\ ,\quad 1\ ,\quad 1+\sqrt{3}\}$$

$$\{-1-\sqrt{3}\ ,\quad -1\ ,\quad -1\ ,\quad 1-\sqrt{3}\ ,\quad -1+\sqrt{3}\ ,\quad 1\ ,\quad 1\ ,\quad 1+\sqrt{3}\}$$

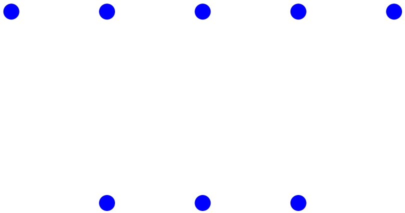
8 vertices,   10 edges,   bipartite

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite

Can the bipartition have parts of unequal size?

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \end{bmatrix}$$

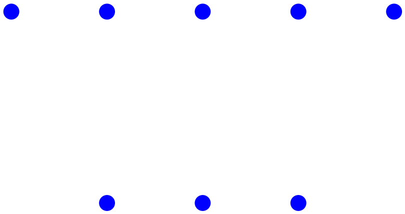


$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite

Can the bipartition have parts of unequal size? NO!

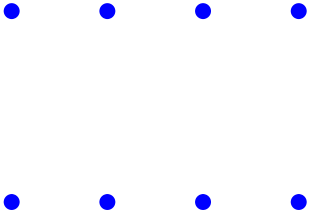
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \end{bmatrix}$$



$$\{-1-\sqrt{3}\,,\;\; -1\,,\;\; -1\,,\;\; 1-\sqrt{3}\,,\;\; -1+\sqrt{3}\,,\;\; 1\,,\;\; 1\,,\;\; 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite with parts of size 4

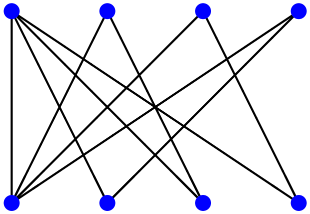
$$\begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\{-1-\sqrt{3}\ ,\quad -1\ ,\quad -1\ ,\quad 1-\sqrt{3}\ ,\quad -1+\sqrt{3}\ ,\quad 1\ ,\quad 1\ ,\quad 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite with parts of size 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

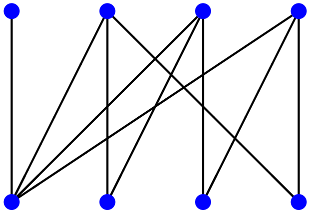


degree sequence $(2, 2, 2, 2, 2, 2, 4, 4)$

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite with parts of size 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$



degree sequence $(1, 2, 2, 2, 3, 3, 3, 4)$

Observation

The degree sequence of a graph is not determined by the adjacency spectrum

Question

Are the sizes of the two parts of a bipartite graph determined by the adjacency spectrum?

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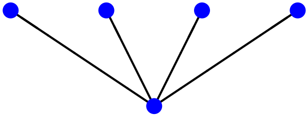
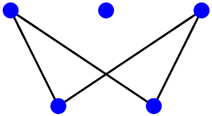
The degree sequence of a graph is not determined by the adjacency spectrum

Question

Are the sizes of the two parts of a bipartite graph determined by the adjacency spectrum?

General answer is NO!





both graphs have adjacency spectrum

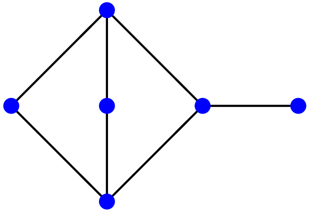
$$\{-2,\ 0,\ 0,\ 0.\ 2\}$$

Theorem (van Dam, WHH 2008)

For every positive integer k there exist two connected bipartite graphs with the same adjacency spectrum for which the sizes of the larger parts of the bipartitions differ by k

- Being connected is not determined by the adjacency spectrum
- Being a tree is not determined by the adjacency spectrum

Laplacian (matrix)



$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

Laplacian spectrum

$$\{0,\ 3-\sqrt{5},\ 2,\ 3,\ 3,\ 3+\sqrt{5}\}$$

$0 = \mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G

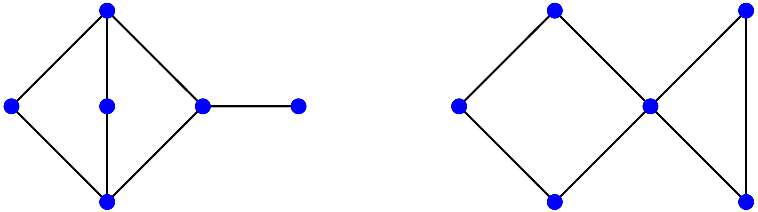
Theorem

• G has $\frac{1}{2}\sum_{i=2}^{n} \mu_i$ edges, and $\frac{1}{n}\prod_{i=2}^{n} \mu_i$ spanning trees

• the number of connected components of G equals the multiplicity of 0

Theorem

G is regular if and only if $n\sum_{i=2}^{n} \mu_i(\mu_i - 1) = (\sum_{i=2}^{n} \mu_i)^2$



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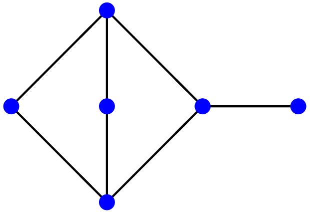

The following properties are **not** determined by the Laplacian spectrum

- number of triangles
- bipartite
- degree sequence

If G is regular of degree k, then $L = kI - A$
hence $\mu_i = k - \lambda_i$ for $i = 1 \ldots n$

Properties determined by one spectrum are also determined by the other spectrum

For regular graphs the following are determined by the spectrum:

- number of vertices, edges, triangles; bipartite
- number of spanning trees, connected components

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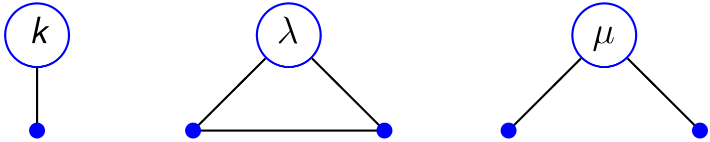
Properties determined by one spectrum are also determined by the other spectrum

For regular graphs the following are determined by the spectrum:

- number of vertices, edges, triangles; bipartite
- number of spanning trees, connected components
- degree sequence
- girth

Strongly regular graph SRG(n, k, λ, μ)



$$A^2 = kI + \lambda A + \mu(J - I - A)$$

$$(A - rI)(A - sI) = \mu J, \quad r + s = \lambda - \mu, \quad rs = \mu - k$$

Every adjacency eigenvalue is equal to k, r, or s

Example SRG$(16, 9, 2, 4)$; Latin square graph

♡	♣	♢	♠
♠	♡	♣	♢
♢	♠	♡	♣
♣	♢	♠	♡

vertices: entries of the Latin square

adjacent: same row, column, or symbol

adjacency spectrum $\{(-3)^6, 1^9, 9\}$

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♠	♢	♣	♡

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Theorem (Shrikhande, Bhagwandas 1965)

G is strongly regular

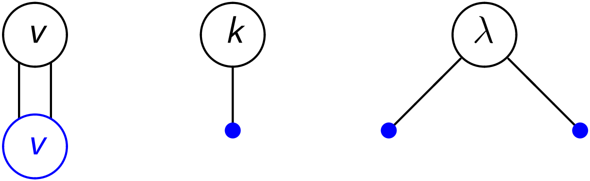
if and only if

G is regular and connected and has exactly three distinct eigenvalues, or G is regular and disconnected with exactly two distinct eigenvalues*

* i.e. G is the disjoint union of $m > 1$ complete graphs of order $\ell > 1$

Incidence graph of a symmetric (v, k, λ)-design

bipartite



Adjacency spectrum

$$\{-k, -\sqrt{k-\lambda}^{(v-1)}, \sqrt{k-\lambda}^{(v-1)}, k\}$$

Example Heawood graph, the incidence graph of the unique symmetric $(7,3,1)$-design (Fano plane)

$$A = \begin{bmatrix} O & N \\ N^\top & O \end{bmatrix} \quad \text{where} \quad N = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spectrum $\{-3,\ -\sqrt{2}^{6},\ \sqrt{2}^{6},\ 3\}$

Theorem (Cvetković, Doob, Sachs 1984)

G is incidence graph of a symmetric (v, k, λ)-design if and only if G has adjacency spectrum

$$\{-k,\ -\sqrt{k-\lambda}^{(v-1)},\ \sqrt{k-\lambda}^{(v-1)},\ k\}$$

Corollary There exists a projective plane of order m if and only if there exists a graph with adjacency spectrum

$$\{-m-1,\ -\sqrt{m}^{m(m+1)},\ \sqrt{m}^{m(m+1)},\ m+1\}$$

For the following properties there exist a pair of cospectral regular graphs where one graph has the property and the other one not

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- being distance-regular of diameter $d \geq 3^*$ ($d \geq 4$ Hoffman 1963, $d = 3$ WHH 1992)

* A distance-regular graphs of diameter 2 is strongly regular

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- having diameter $d \geq 2$ (WHH, Spence 1995)
- having a perfect matching ($\frac{n}{2}$ disjoint edges) (Blázsik, Cummings, WHH 2015)
- NP-hard properties (chromatic number, clique number)

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Characterizations from the spectral point of view

Proposition G has two distinct adjacency eigenvalues if and only if G is the disjoint union of complete graphs having the same order $m > 1$

Proposition G has two distinct Laplacian eigenvalues if and only if G is the disjoint union of complete graphs having the same order $m > 1$, possibly extended with some isolated vertices

Can we characterize the graphs with three distinct adjacency eigenvalues?

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If the graphs are regular and connected, then they are precisely the connected strongly regular graphs

If regularity is not assumed, then there exist other examples, but no characterization known

Theorem (van Dam, WHH 1998)

A connected graph G has three distinct Laplacian eigenvalues if and only if μ and $\overline{\mu}$ are constant

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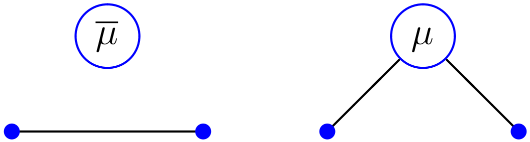

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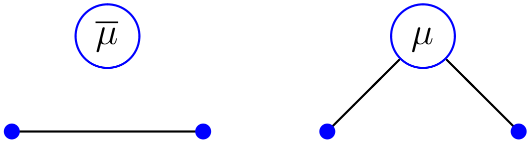
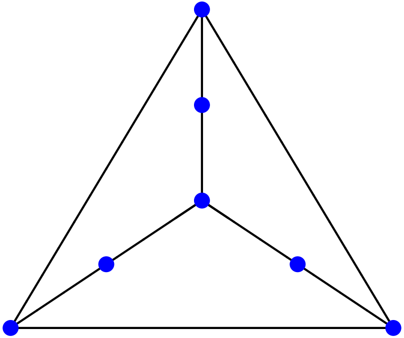


If G is regular of degree k, then $\overline{\mu} = n - 2k + \lambda$, and G is an SRG(n, k, λ, μ)

Example $n = 7,\ \mu = 1,\ \overline{\mu} = 2$



Laplacian spectrum $\{0,\ 3 - \sqrt{2}^{\,3},\ 3 + \sqrt{2}^{\,3}\}$

Theorem (Cameron, Goethals, Seidel, Shult 1976)

A graph G has least adjacency eigenvalue ≥ -2 if and only if G is a generalized line graph, or G belongs to a finite set of exceptional graphs ($n \leq 36$)

Book: Spectral generalisations of line graphs,
Cvetković, Rowlinson, Simić 2004

Proposition G has least adjacency eigenvalue ≥ -1 if and only if G is the disjoint union of complete graphs

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Proof 1 $A + I$ is positive semi-definite, so it is the Gram matrix of a set of unit vectors with inner product 0 or 1

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Proof 1 $A + I$ is positive semi-definite, so it is the Gram matrix of a set of unit vectors with inner product 0 or 1

Proof 2 The path $P_3 =$  has spectrum $\{-\sqrt{2},\ 0,\ \sqrt{2}\}$, and by interlacing it can not be an induced subgraph of G

Proposition The following are equivalent:

- $G = K_{n_1} + \ldots + K_{n_m}$
- G has adjacency spectrum

$$\{-1^{n-m},\ n_1 - 1, \ldots, n_m - 1\}$$

- G has Laplacian spectrum

$$\{0^m,\ n_1^{n_1 - 1}, \ldots, n_m^{n_m - 1}\}$$

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- $G = K_{n_1} + \ldots + K_{n_m}$
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- G has Laplacian spectrum

$$\{0^m,\ n_1^{n_1-1}, \ldots, n_m^{n_m-1}\}$$

$K_{n_1} + \ldots + K_{n_m}$ is determined by the adjacency spectrum and by the Laplacian spectrum

Question (van Dam, WHH 2003)

Which graphs are determined by their spectrum?

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Which graphs are determined by their spectrum?

What fraction of graphs on n vertices is determined by the adjacency spectrum?

What fraction of graphs on n vertices is determined by the Laplacian spectrum?

Conjecture 1 Almost all graphs are determined by the adjacency spectrum

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Conjecture 2 Almost all graphs are determined by the Laplacian spectrum

Conjecture 1 Almost all graphs are determined by the adjacency spectrum

Conjecture 2 Almost all graphs are determined by the Laplacian spectrum

Conjecture 3 Almost all graphs are determined by the generalized spectrum*

* the adjacency spectrum together with the adjacency spectrum of the complement

n	number of graphs	A	L	A & $\overline{A}$
1	1	1	1	1
2	2	1	1	1
3	4	1	1	1
4	11	1	1	1
5	34	0.941	1	1
6	156	0.936	0.934	1
7	1044	0.895	0.875	0.962
8	12346	0.861	0.857	0.906
9	274668	0.814	0.845	0.840
10	12005168	0.787	**0.882**	0.799
11	1018997864	**0.789**	**0.910**	0.792
12	165091172592	**0.812**	**0.940**	$\geq$ **0.812**

Fractions of graphs of order n determined by the spectrum

Approach to Conjecture 1

Find all orthogonal matrices Q such that $Q^\top A Q$ is a (0,1)-matrix

If each Q is a permutation matrix, then the graph is determined by the adjacency spectrum

Approach to Conjecture 3

Find all regular orthogonal matrices Q such that $Q^\top A Q$ is a (0,1)-matrix

If each Q is a permutation matrix, then the graph is determined by the generalized spectrum

G is controllable if $W = \begin{bmatrix} \mathbf{1} & A\mathbf{1} & A^2\mathbf{1} & \dots & A^{n-1}\mathbf{1} \end{bmatrix}$ is nonsingular

Approach (Wang, Xu 2006)

If G is controllable and $Q^\top AQ$ is a (0,1)-matrix A' for some regular orthogonal matrix Q, then

- Q is unique (for fixed A')
- Q is rational
- SNF(W) gives an integer ℓ such that ℓQ is integral
- Often $\ell = 2$ in which case Q is characterized

Support for Conjecture 3

- The known fractions for $n = 1, \ldots, 12$

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- (Wang 2014) Graphs for which $\det W/2^{\lfloor n/2 \rfloor}$ is odd and square free are determined by the generalized spectrum

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- The known fractions for $n = 1, \ldots, 12$
- (Wang, Xu 2006) Many randomly generated graphs turn out to be determined by the generalized spectrum
- (Wang 2014) Graphs for which $\det W/2^{\lfloor n/2 \rfloor}$ is odd and square free are determined by the generalized spectrum
- (O'Rourke, Touri 2016) Almost all graphs are controllable